Fragmentation of fractal random structures

Martin Weigel

Applied Mathematics Research Centre, Coventry University, Coventry, United Kingdom and
Institut für Physik, Johannes Gutenberg-Universität Mainz, Germany

with Eren Metin Elçi and Nikolaos G. Fytas (Coventry)

International Conference on Computer Simulation in Physics and Beyond
Moscow, September 7, 2015

see Phys. Rev. Lett. 114, 115701 (2015) and arXiv:1509.00668

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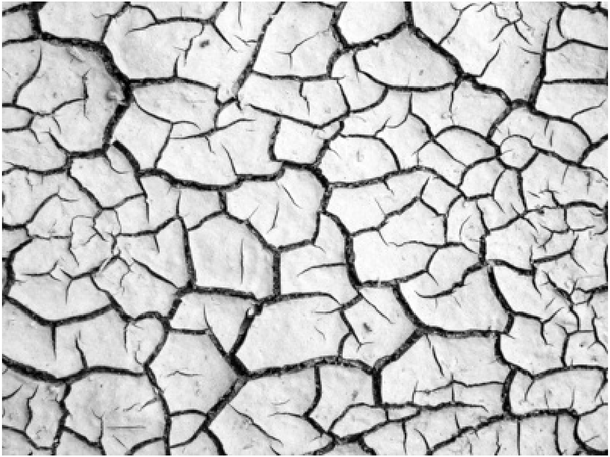

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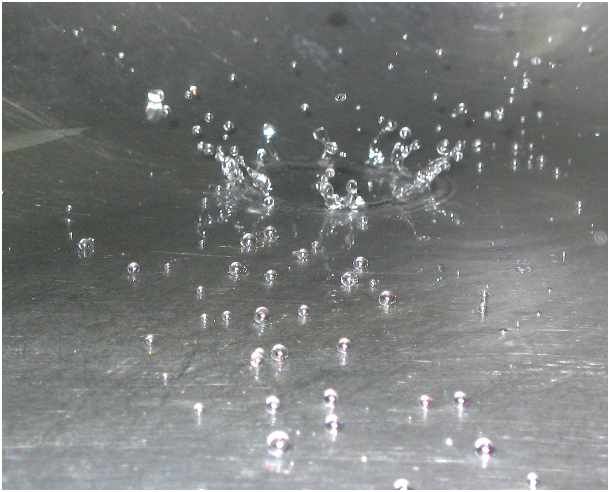

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$a(s)$	fragmentation rate of clusters of size s
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This leads to a mean-field description with scaling properties.

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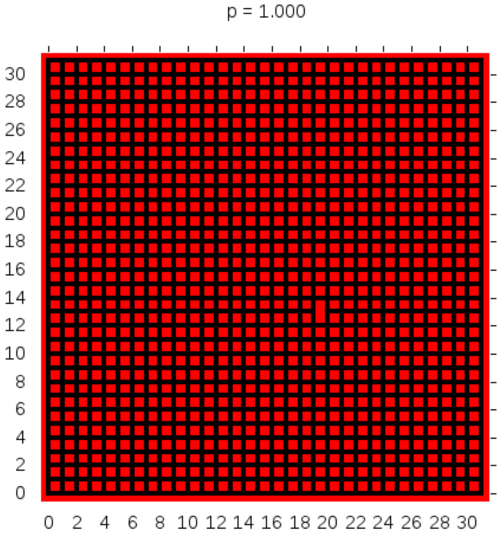
- equivalent to Potts model for integer q
- includes percolation ($q \to 1$), Ising model ($q = 2$), random resistor networks ($q \to 0$), ...
- continuous percolation transition for $q < q_c$
- first-order percolation transition for $q > q_c$
- $q_c = 4$ in 2D, $q_c \approx 2.8$ in 3D

Bond fragmentation



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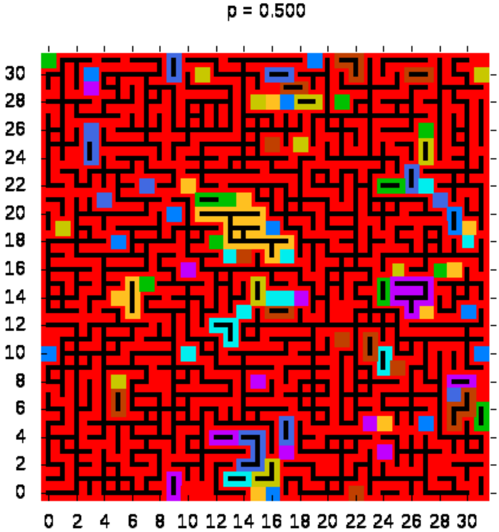

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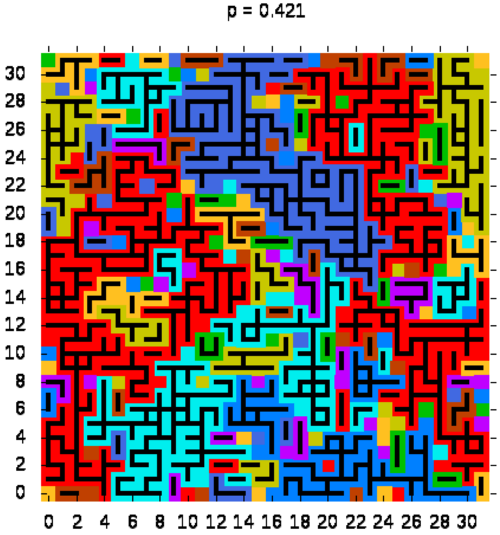

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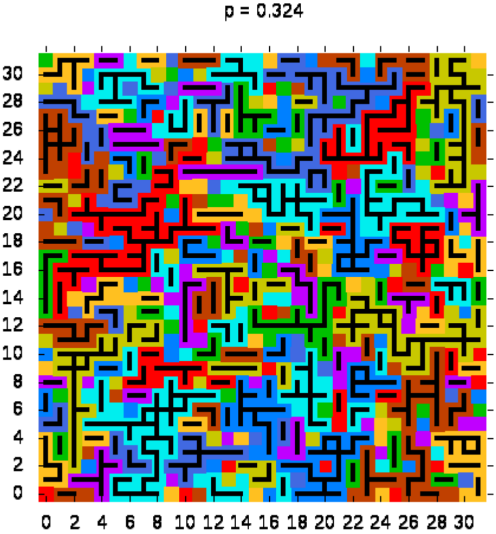

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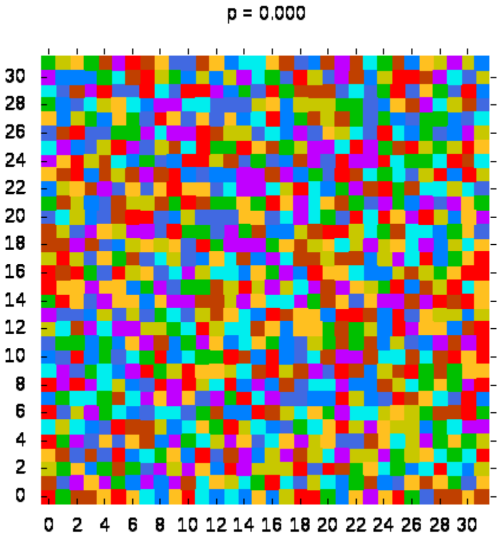

Bridges and non-bridges

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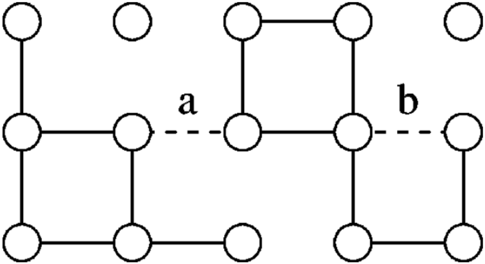

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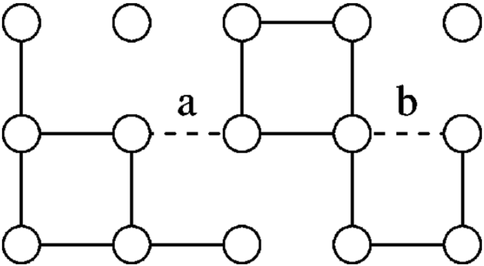
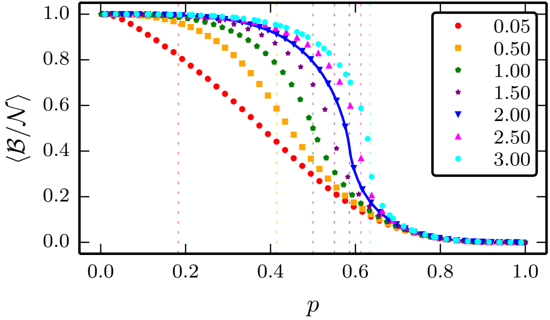


Fragmentation rate effected by breaking bonds randomly at a constant rate depends on the density of bridges.

Bridge density

(Relative) bridge density as a function of p:



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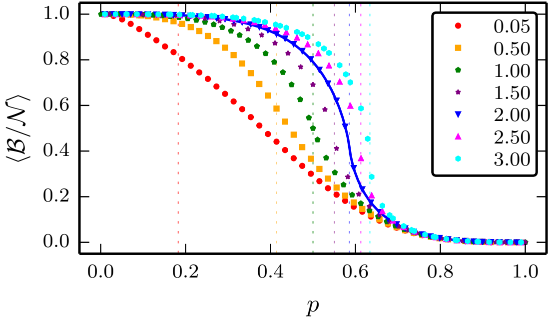

Critical point $p_c = \sqrt{q}/(1+\sqrt{q})$ is where the change in bridge density and hence the fragility becomes singular (for $q \geq 2$).

Scaling theory

How do the fragments look like? Scaling theory for $a(s)$ and $b(s, s')$ for percolation (`Gyure and Edwards, 1992`):

$$\begin{aligned} a(s) &\sim s^{\lambda} \\ b(s, s') &\sim s^{-\phi} g(s'/s) \end{aligned} \tag{3}$$

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Are λ and ϕ related to the known percolation exponents?

Bridge-edge identity

Based on the Russo-Margulis formula in percolation theory, it is possible to prove the following **bridge-edge identity**

$$\langle \mathcal{B} \rangle = \frac{\langle \mathcal{N} \rangle - p}{(1-p)(1-q)} \tag{5}$$

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Clearly, the case $q \to 1$ is singular. There one finds

$$\langle \mathcal{B} \rangle = \frac{1}{p-1} \lim_{q \to 1} \frac{\mathrm{d}}{\mathrm{d}q} \langle \mathcal{N} \rangle$$

Bridge densities

Equation (5) holds for arbitrary graphs. At criticality, $\langle \mathcal{N} \rangle = 1/2$ and hence

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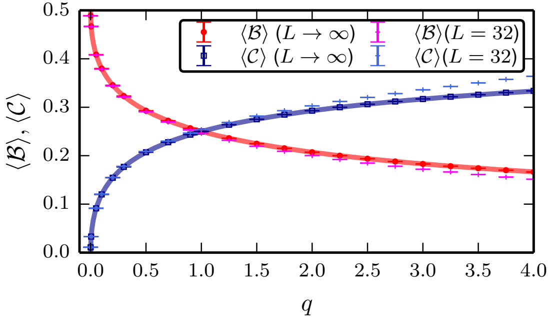

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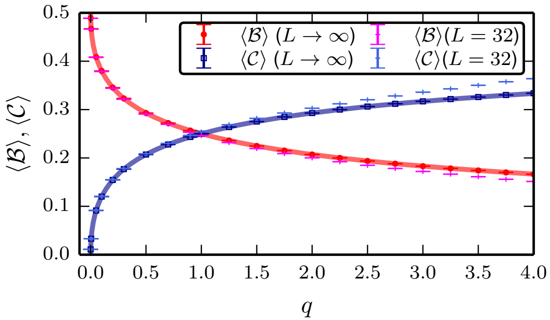

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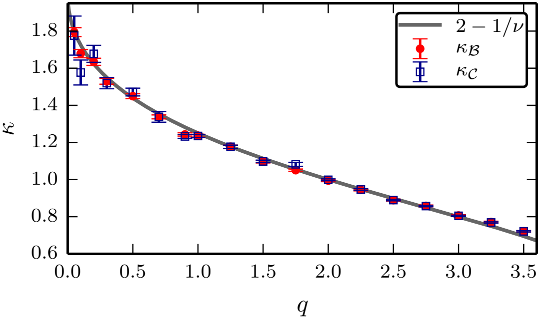

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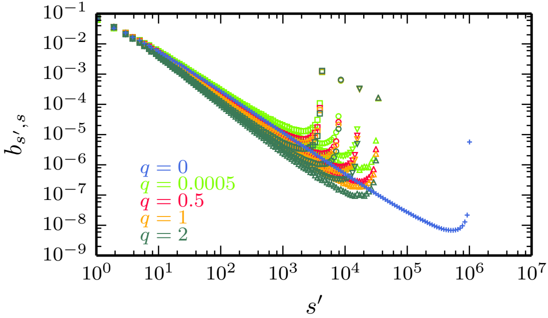

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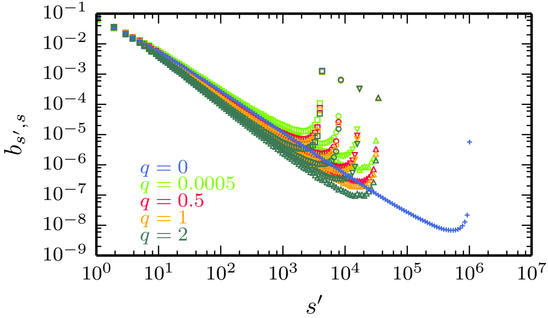

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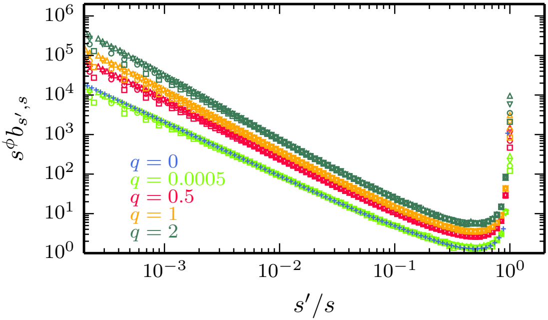

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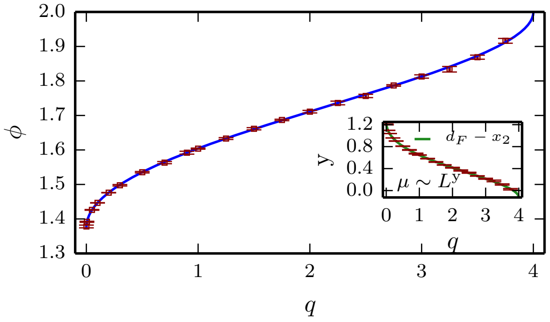

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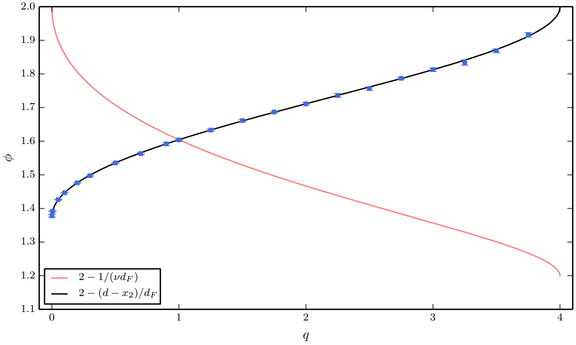

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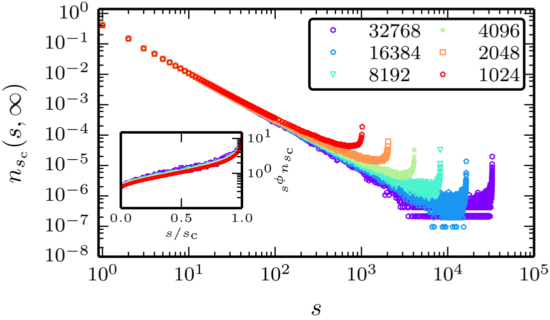
- introduce cutoff to regularize this and model milling or fluid break-up

Start from a fractal, critical configuration and remove bonds at random. Do not break up any clusters smaller than s_c.



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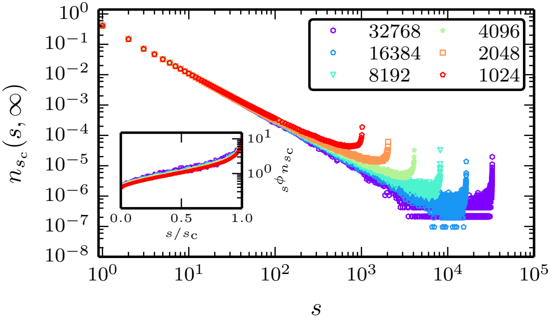

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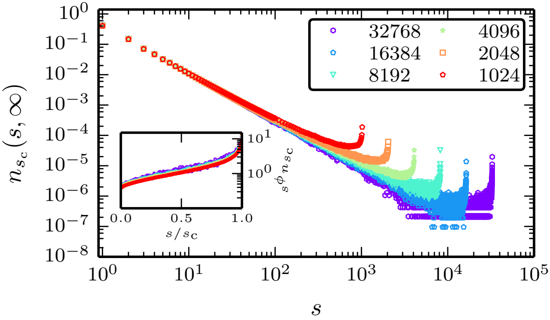

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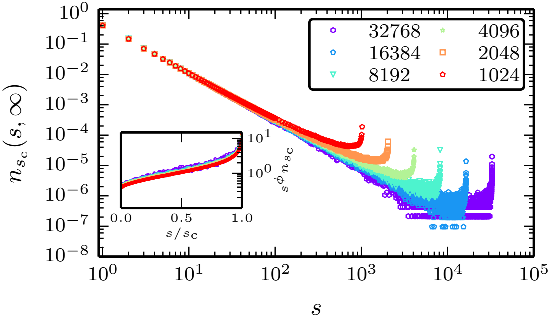

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$$n_{s_c}(s,\infty) \sim s^{-\chi}\mathcal{F}\left(\frac{s}{s_c}\right). \tag{9}$$

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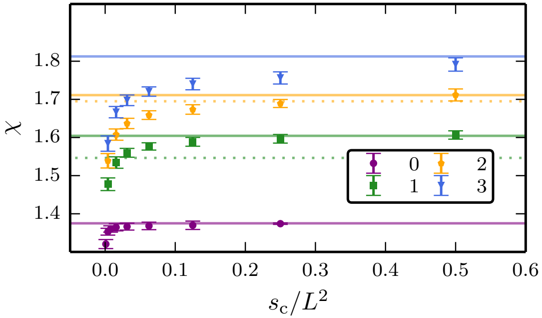

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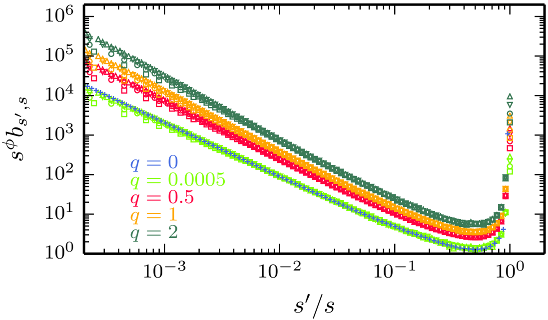

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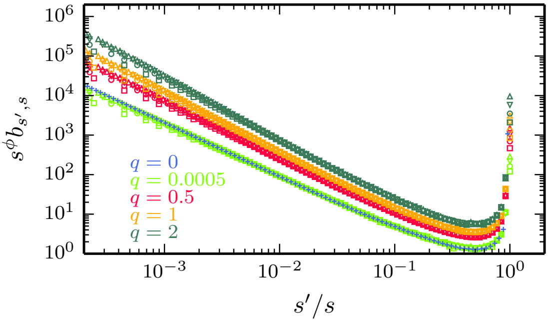

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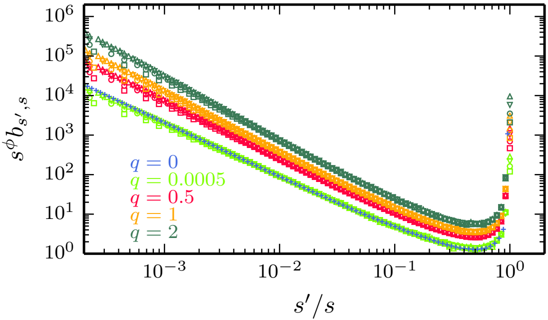

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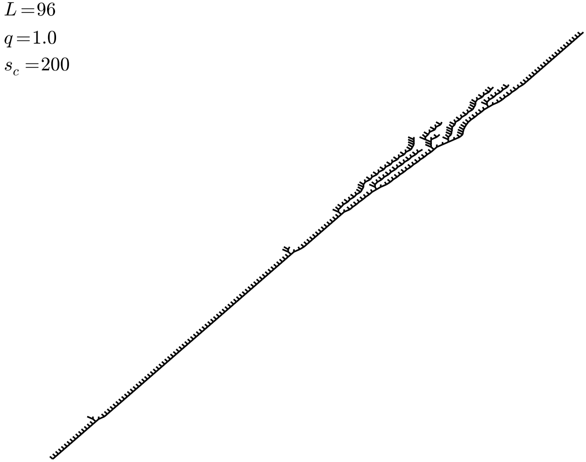

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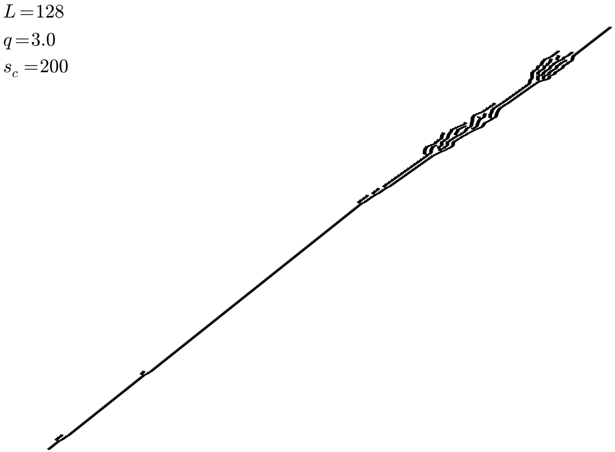

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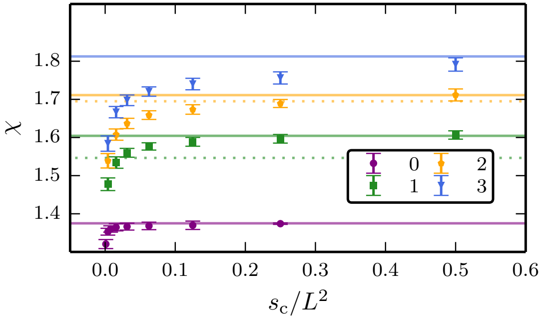

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- can use bridge-free clusters to determine backbone fractal dimension

Eren M. Elçi, MW, Phys. Rev. E 88, 033303 (2013).
Eren M. Elçi, MW, Nikolaos G. Fytas, Phys. Rev. Lett. **114**, 115701 (2015)
Eren M. Elçi, MW, Nikolaos G. Fytas, *Bridges in the random-cluster model*, arXiv:1509.00668.

Conclusions

Summary:

- full characterization of scaling behavior of fragmentation of critical random-cluster model
- new exact relation between bridge and bond densities
- dynamical, non-equilibrium fragmentation with a cutoff leads to power-law fragment distributions, described by equilibrium exponents
- relevant to experiment (universality)?

Extensions:

- our results extend to three dimensions
- vertex fragmentation related to multi-arm exponents
- can use bridge-free clusters to determine backbone fractal dimension
- dynamical fragmentation of solid objects gives different set of exponents

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